A Reminder on Home Test 2

Please be reminded that Home Test 2 will be held **next Saturday (05/12/2020)**.

- Duration: 24 hours (05/12 12:00 noon to 06/12 12:00 noon)
- Content: Section 6 - 8 of Lecture Note. (Reference: Chapter 4 and 5 of textbook)
- Delivery: The test paper will be sent to the **university email account** at 12:00 noon.
- Submission: Submit **one PDF file** to **Blackboard**. (Same as homework assignments)

A review exercise is posted. The suggest solution will be posted on 30/11/2020 (Monday).

Continuous Functions on Intervals

In this section, we study continuous functions defined on intervals. There are four important results. The first two theorems rely on the fact that closed bounded intervals are **compact**. The last two theorems rely on the fact that intervals are **connected**.

Boundedness Theorem (c.f. 5.3.2)**.** Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I. i.e., there exists $M > 0$ such that

$$|f(x)| \leq M, \quad \forall x \in I.$$

Definition (c.f. Definition 5.3.3)**.** Let $f : A \to \mathbb{R}$ be a function. f is said to have an *absolute maximum* on A if there exists $x^* \in A$ such that

$$f(x^*) \geq f(x), \quad \forall x \in A.$$

Similarly, f is said to have an *absolute mimimum* on A if there exists $x_* \in A$ such that

$$f(x_*) \leq f(x), \quad \forall x \in A.$$

Maximum-Minimum Theorem (c.f. 5.3.4)**.** Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.

Remark. In the Lecture Note, closed bounded intervals are replaced by compact subsets. Nonetheless, the ideas are similar. The textbook tends to avoid the term compactness.

Example 1. Consider the continuous function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = 1/x$. Notice that f is unbounded on $(0, 1]$ and hence does not have an absolute maximum on $(0, 1]$. It follows that the **Boundedness Theorem** and the **Maximum-Minimum Theorem** do not hold if the assumption that the interval I being closed is dropped.

Exercise. Show that f is unbounded on $(0, 1]$.

Example 2. Consider the continuous function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$. Notice that f is unbounded on $[0, \infty)$ and hence does not have an absolute maximum on $[0, \infty)$. It follows that the **Boundedness Theorem** and the **Maximum-Minimum Theorem** do not hold if the assumption that the interval I being bounded is dropped.

Exercise. Show that f is unbounded on $[0, \infty)$.

Example 3 (c.f. Section 5.3, Ex.13)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose

$$\lim_{x\to-\infty} f(x) = L = \lim_{x\to\infty} f(x).$$

Show that

(a) f is bounded on $\mathbb{R}$.

(b) f has either an absolute maximum or an absolute minimum.

Solution. .

(a) By definition of limits to infinity, there exists $a < 0$ and $b > 0$ such that

$$|f(x) - L| < 1 \quad \text{whenever } x < a \text{ or } x > b.$$

i.e., $L - 1 < f(x) < L + 1$ whenever $x \in (-\infty, a) \cup (b, \infty)$. Now, notice that f is continuous on the closed bounded interval $[a, b]$. By the **Boundedness Theorem**, there exists $M' > 0$ such that $|f(x)| < M'$ whenever $x \in [a, b]$. We can then see that f is bounded by $M = \max\{|L \pm 1|, M'\}$ on $\mathbb{R}$.

Exercise. Show that if $\alpha \le x \le \beta$, then $|x| \le \max\{|\alpha|, |\beta|\}$.

(b) Firstly, suppose f is a constant function. Then we must have $f(x) = L$ for all $x \in \mathbb{R}$. In this case, f must have absolute maximum and minimum. On the other hand, suppose f is not a constant function. Then there exists some $c \in \mathbb{R}$ such that $f(c) \ne L$. By definition of limits to infinity, there exists $a < c$ and $b > c$ such that

$$|f(x) - L| < |f(c) - L| \quad \text{whenever } x < a \text{ or } x > b. \tag{1}$$

Now, notice that f is continuous on the closed bounded interval $[a, b]$. By the **Maximum-Minimum Theorem**, there exists $x^*, x_* \in [a, b]$ such that

$$f(x_*) \le f(x) \le f(x^*), \quad \forall x \in [a, b].$$

In particular, $f(x_*) \le f(c) \le f(x^*)$. To expand the absolute value in (1), consider the case $f(c) > L$, then whenever $x \in (-\infty, a) \cup (b, \infty)$,

$$f(x) - L \le |f(x) - L| < f(c) - L \implies f(x) < f(c).$$

In this case, $f(x) \le f(x^*)$ for all $x \in \mathbb{R}$. i.e, f has an absolute maximum on $\mathbb{R}$. Conversely, consider the case $f(x) < L$, then whenever $x \in (-\infty, a) \cup (b, \infty)$,

$$L - f(x) \le |f(x) - L| < L - f(c) \implies f(c) < f(x).$$

In this case, $f(x_*) \le f(x)$ for all $x \in \mathbb{R}$. i.e, f has an absolute mimimum on $\mathbb{R}$.

Bolzanos Intermediate Value Theorem (c.f. 5.3.7)**.** Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists a point $c \in I$ between a and b such that $f(c) = k$.

The following theorem characterise whether a subset of $\mathbb{R}$ is an interval or not. Using this characterisation and the **Intermediate Value Theorem**, we can prove the fourth theorem.

Characterization Theorem (c.f. 2.5.1)**.** If $S \subseteq \mathbb{R}$ contains at least two points and has the property

$$x, y \in S \implies [x, y] \subseteq S,$$

then S is an interval.

Preservation of Intervals Theorem (c.f. 5.3.10)**.** Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.

Example 4. Consider $A = [0, 1] \cup [2, 3]$ and the continuous function $f : A \to \mathbb{R}$ defined by $f(x) = x$. Notice that $f(A) = A$ is not an interval. Moreover, $f(0) < 1.5 < f(3)$ but there are no values $x \in A$ satisfies $f(x) = 1.5$. It follows that the **Intermediate Value Theorem** and the **Preservation of Intevals Theorem** do not hold if the assumption that the domain being an interval is dropped.

Exercise. Show that f is continuous on $[0, 1] \cup [2, 3]$.

Example 5. Consider the polynomial $f(x) = x^5 + x^3 + 1$, which is continuous on $\mathbb{R}$. We proceed to find a root of f by using the **Bisection Method**.

- Set $a_1 = -1$ and $b_1 = 1$. Notice that $f(a_1) = -1 < 0$ and $f(b_1) = 3 > 0$. Hence by the **Intermediate Value Theorem**, f has a root in the interval $[a_1, b_1]$. Consider the mid-point $c_1 = (a_1 + b_1)/2 = 0$, we calculate $f(c_1) = f(0) = 1 > 0$.

- Set $a_2 = a_1$ and $b_2 = c_1$. Notice that $f(a_2) = -1 < 0$ and $f(b_2) = 1 > 0$. Hence by the **Intermediate Value Theorem**, f has a root in the interval $[a_2, b_2]$. Consider the mid-point $c_2 = (a_2 + b_2)/2 = -0.5$, we calculate $f(c_2) = f(-0.5) = 0.84375 > 0$.

- Similarly for each $n \in \mathbb{N}$, if $f(c_n) < 0$, set $a_{n+1} = c_n$ and $b_{n+1} = b_n$; otherwise set $a_{n+1} = a_n$ and $b_{n+1} = c_n$. Then the values of f at the endpoints a_{n+1} and b_{n+1} have different signs. By the **Intermediate Value Theorem**, f has a root in the interval $[a_{n+1}, b_{n+1}]$.

We have constructed a nested sequence of closed bounded intervals $[a_n, b_n]$, each contains a root of f, with an additional property that $\lim(b_n - a_n) = 0$. Hence the value of a root of f can be approximated by their mid-points c_n, with the error converging to zero.

Remark. The **Bisection Method** allow us to numerically approximate real solutions of continuous functions, it may not give us the exact value of the solutions. Moreover, this method may not give us all real solutions.